

# Distributed-order fractional wave equation on a finite domain. Stress relaxation in a rod

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## Abstract

We study waves in a rod of finite length with a viscoelastic constitutive equation of fractional distributed-order type for the special choice of weight functions. Prescribing boundary conditions on displacement, we obtain case corresponding to stress relaxation. In solving system of differential and integro-differential equations we use the Laplace transformation in the time domain.

**Keywords:** fractional derivative, distributed-order fractional derivative, fractional viscoelastic material, distributed-order wave equation, stress relaxation

## 1 Introduction

Fractional derivatives have been used in describing physical phenomena such as viscoelasticity, diffusion and wave phenomena. There are two approaches in formulating differential equations with fractional derivatives in physics and mechanics. In the first approach classical "integer order" differential equations of a process are modified by introducing fractional derivatives instead of integer order ones (see books by Mainardi 1997, Podlubny 1999, and Kilbas *et al.* 2006). In the second approach one uses variational principles such as the Hamilton principle as a starting point for deriving equations of a process, where a modification of the classical case is achieved by replacing some (or all) integer order derivatives in Lagrangian density by fractional derivatives of certain kind. Then the resulting Euler-Lagrange equations are equations of a process and they contain both left and right fractional derivatives (see papers by Agrawal 2002, Atanackovic & Stankovic 2007, Atanackovic *et al.* 2008).

In this paper we generalize classical wave equation for one-dimensional elastic body by following the first approach. Recall the classical setting. Consider the equation of motion

$$\frac{\partial}{\partial x} \sigma(x, t) = \rho \frac{\partial^2}{\partial t^2} u(x, t), \quad x \in [0, L], \quad t > 0, \quad (1.1)$$

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where  $\rho$ ,  $\sigma$  and  $u$  denote density, stress and displacement of a material at a point positioned at  $x$  and at a time  $t$ , respectively. It is coupled with the Hooke Law

$$\sigma(x, t) = E\mathcal{E}(x, t), \quad x \in [0, L], \quad t > 0, \quad (1.2)$$

where  $E$  is a modulus of elasticity and  $\mathcal{E}$  is a strain measure, defined by

$$\mathcal{E}(x, t) = \frac{\partial}{\partial x} u(x, t), \quad x \in [0, L], \quad t > 0. \quad (1.3)$$

Combining (1.1) - (1.3), the classical wave equation is obtained as

$$\frac{\partial^2}{\partial x^2} u(x, t) = \frac{\rho}{E} \frac{\partial^2}{\partial t^2} u(x, t), \quad x \in [0, L], \quad t > 0.$$

We propose the generalization of a constitutive equation (1.2) by replacing it with a constitutive equation which corresponds to a generalized viscoelastic body:

$$\int_0^1 \phi_1(\alpha) {}_0D_t^\alpha \sigma(x, t) d\alpha = E \int_0^1 \phi_2(\alpha) {}_0D_t^\alpha \mathcal{E}(x, t) d\alpha, \quad x \in [0, L], \quad t > 0, \quad (1.4)$$

where  $E$  is a positive constant (having dimension of stress),  $\phi_1$  and  $\phi_2$  are given functions or distributions and  ${}_0D_t^\alpha y$  is the left Riemann-Liouville fractional derivative of a function  $y \in AC([0, T])$ , for every  $T > 0$ , of the order  $\alpha \in [0, 1]$ , defined as

$${}_0D_t^\alpha y(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{y(\tau)}{(t-\tau)^\alpha} d\tau, \quad t > 0,$$

where  $\Gamma$  is the Euler gamma function. Recall,  $AC([0, T])$  denotes the space of absolutely continuous functions (for a detailed account on fractional calculus see a book Samko *et al.* 1993). In case when  $\phi_1$  and  $\phi_2$  are distributions, we assume that  $\phi_1$  and  $\phi_2$  are compactly supported by  $[0, 1]$  ( $\phi_1, \phi_2 \in \mathcal{E}'(\mathbb{R})$ ,  $\text{supp } \phi_1, \text{supp } \phi_2 \subset [0, 1]$ ). In this case integrals in (1.4) are defined as

$$\left\langle \int_{\text{supp } \phi} \phi(\alpha) {}_0D_t^\alpha h(t) d\alpha, \varphi(t) \right\rangle := \langle \phi(\alpha), \langle {}_0D_t^\alpha h(t), \varphi(t) \rangle \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

For details see a work by Atanackovic *et al.* (2009a). Recall,  $\mathcal{D}'_+(\mathbb{R})$  denotes the space of distributions supported by  $[0, \infty)$  and  $\langle h(t), \varphi(t) \rangle$  denotes the action of a distribution  $h \in \mathcal{D}'_+(\mathbb{R})$  on a test function  $\varphi \in \mathcal{D}(\mathbb{R})$  (see a book by Vladimirov 1984).

In (1.4),  $\phi_1$  and  $\phi_2$  denote constitutive functions or distributions that are determined experimentally. The constitutive equations of type (1.4) were used earlier in papers by Hartley & Lorenzo (2003), Atanackovic (2002b), Atanackovic *et al.* (2009b) and Atanackovic *et al.* (2009c). There are number of forms that  $\phi_1$  and  $\phi_2$  can take (see a paper by Hartley & Lorenzo 2003, for example). In the sequel we assume that

$$\phi_1(\alpha) := a^\alpha, \quad \phi_2(\alpha) := b^\alpha, \quad \alpha \in (0, 1), \quad a \leq b. \quad (1.5)$$

The restriction  $a \leq b$  follows from the Second Law of Thermodynamics (see for example papers by Atanackovic 2002a and Atanackovic 2003). If  $a = b$ , then (1.4) reduces to the Hooke Law. The choice of  $\phi_1$  and  $\phi_2$  in the form (1.5) is the simplest choice guaranteeing dimensional homogeneity.

Note that with  $\phi_1(\mu) := \delta(\mu) + \tau_\epsilon^\alpha \delta(\mu - \alpha)$  and  $\phi_2(\mu) := E_\infty \tau_\epsilon^\beta \delta(\mu - \beta)$  ( $\delta$  denotes the Dirac distribution) we obtain

$$\sigma + \tau_{\epsilon 0}^\alpha D_t^\alpha \sigma = E_\infty \tau_{\epsilon 0}^\beta D_t^\beta \epsilon, \quad (1.6)$$

while with  $\phi_1(\mu) := \delta(\mu) + \tau_\epsilon^\alpha \delta(\mu - \alpha)$  and  $\phi_2(\mu) := E_0(\delta(\mu) + \tau_\sigma^\alpha \delta(\mu - \alpha) + \tau_\sigma^\beta \delta(\mu - \beta))$  we obtain

$$\sigma + \tau_{\epsilon 0}^\alpha D_t^\alpha \sigma = E_0 \left( 1 + \tau_{\sigma 0}^\alpha D_t^\alpha + \tau_{\sigma 0}^\beta D_t^\beta \right) \epsilon. \quad (1.7)$$

Recall, system (1.1), (1.3) and (1.6), respectively system (1.1), (1.3) and (1.7), was treated in a work by Rossikhin & Shitikova (2001b), respectively in a work by Rossikhin & Shitikova (2001a). Also note that the distributed order dissipation of type (1.4) was also used in the context of one degree of freedom mechanical systems in papers by Atanackovic *et al.* (2005) and Atanackovic & Pilipovic (2005).

Our aim is to find functions  $u$  and  $\sigma$ , locally integrable on  $\mathbb{R}$  and equal to zero for  $t < 0$ , so that these functions satisfy (1.1), (1.3), (1.4), for  $x \in [0, L]$  and  $t > 0$ , as well as the appropriate initial and boundary conditions. Actually, we will introduce dimensionless quantities and transform the system (1.1), (1.3), (1.4), subject to (2.1) and (2.2), into the system (2.3), subject to (2.4) and (2.5).

The paper is organized as follows. In §2 we introduce dimensionless quantities, proceed by formal calculation and by the use of the Laplace transformation we obtain solutions to (1.1), (1.3), (1.4) in the convolution form. We impose initial conditions as well as boundary conditions to (1.1), (1.3), (1.4). Boundary conditions describe a rod that is fixed at one of its ends, while the other end is subject to a prescribed displacement  $\Upsilon$  (this is the case of stress relaxation if  $\Upsilon = \Upsilon_0 H$ , with  $H$  being the Heaviside function). Section 3 is devoted to the calculation of the inverse Laplace transformation, which leads to the explicit form of a solution. More precisely, we investigate some properties of functions in order to be able to apply the Cauchy residues theorem, which is used to calculate the inverse Laplace transformation. We obtain displacement  $u$  and stress  $\sigma$  for the boundary condition  $\Upsilon = \Upsilon_0 H$  in §3.1.1, as well as for  $\Upsilon = \Upsilon_0 H + F$ , where  $F$  is an appropriate function supported by  $[0, \infty)$ , in §3.1.2. We conclude that solutions are locally integrable functions supported by  $[0, \infty)$ . Moreover, they are smooth functions for  $t > 0$ . Numerical examples corresponding to stress relaxation are presented in §4. Concluding remarks are given in §5.

## 2 Convolution form of solutions. Formal calculation.

We prescribe initial conditions for system (1.1), (1.3), (1.4)

$$u(x, 0) = 0, \quad \frac{\partial}{\partial t} u(x, 0) = 0, \quad \sigma(x, 0) = 0, \quad \mathcal{E}(x, 0) = 0, \quad x \in [0, L]. \quad (2.1)$$

We subject system (1.1), (1.3), (1.4) to boundary conditions corresponding to case of stress relaxation

$$u(0, t) = 0, \quad u(L, t) = \Upsilon(t), \quad t \in \mathbb{R}. \quad (2.2)$$

Function  $\Upsilon$  is locally integrable function equal to zero for  $t < 0$ .

Introducing dimensionless quantities

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{L\sqrt{\frac{p}{E}}}, \quad \bar{u} = \frac{u}{L}, \quad \bar{\sigma} = \frac{\sigma}{E}, \quad \bar{\Upsilon} = \frac{\Upsilon}{L}, \quad \bar{\phi}_1 = \frac{\phi_1}{(L\sqrt{\frac{p}{E}})^\alpha}, \quad \bar{\phi}_2 = \frac{\phi_2}{(L\sqrt{\frac{p}{E}})^\alpha},$$

and using the fact that the fractional derivative transforms as

$${}_0D_t^\alpha u(t) = \left( L \sqrt{\frac{\rho}{E}} \right)^\alpha {}_0D_t^\alpha u(t),$$

we obtain, after omitting bar over dimensionless quantities, the following system

$$\begin{aligned} \frac{\partial}{\partial x} \sigma(x, t) &= \frac{\partial^2}{\partial t^2} u(x, t), \\ \int_0^1 \phi_1(\alpha) {}_0D_t^\alpha \sigma(x, t) d\alpha &= \int_0^1 \phi_2(\alpha) {}_0D_t^\alpha \mathcal{E}(x, t) d\alpha, \\ \mathcal{E}(x, t) &= \frac{\partial}{\partial x} u(x, t), \quad x \in [0, 1], \quad t > 0. \end{aligned} \quad (2.3)$$

System (2.3) is subject to initial

$$u(x, 0) = 0, \quad \frac{\partial}{\partial t} u(x, 0) = 0, \quad \sigma(x, 0) = 0, \quad \mathcal{E}(x, 0) = 0, \quad x \in [0, 1], \quad (2.4)$$

and boundary conditions

$$u(0, t) = 0, \quad u(1, t) = \Upsilon(t), \quad t \in \mathbb{R}. \quad (2.5)$$

Additional assumptions on  $\Upsilon$  will be given in §3.1.1 and §3.1.2.

In the sequel we assume that  $\phi_1$  and  $\phi_2$  are functions given by (1.5). Using (1.5) in (2.3)<sub>2</sub> and formally applying the Laplace transformation to (2.3) and (2.4), we obtain

$$\begin{aligned} \frac{\partial}{\partial x} \tilde{\sigma}(x, s) &= s^2 \tilde{u}(x, s), \\ \tilde{\sigma}(x, s) \int_0^1 (as)^\alpha d\alpha &= \tilde{\mathcal{E}}(x, s) \int_0^1 (bs)^\alpha d\alpha, \\ \tilde{\mathcal{E}}(x, s) &= \frac{\partial}{\partial x} \tilde{u}(x, s), \quad x \in [0, 1], \quad s \in D. \end{aligned} \quad (2.6)$$

Recall, the Laplace transformation of  $f \in L_{loc}^1(\mathbb{R})$ ,  $f \equiv 0$  in  $(-\infty, 0]$  and  $|f(t)| \leq ce^{at}$ ,  $t > 0$ , for some  $a > 0$ , is defined by

$$\tilde{f}(s) = \mathcal{L}[f(t)](s) := \int_0^\infty f(t) e^{-st} dt, \quad \operatorname{Re} s > a$$

and analytically continued into the appropriate domain  $D$ . Domain  $D$  for (2.6) is determined after (2.7), bellow.

System (2.6) reduces to

$$\frac{\partial^2}{\partial x^2} \tilde{u}(x, s) - s^2 \frac{\ln(bs)}{\ln(as)} \frac{as - 1}{bs - 1} \tilde{u}(x, s) = 0, \quad x \in [0, 1], \quad s \in D,$$

whose formal solution is

$$\tilde{u}(x, s) = C_1(s) e^{xs \sqrt{\frac{\ln(bs)}{\ln(as)} \frac{as - 1}{bs - 1}}} + C_2(s) e^{-xs \sqrt{\frac{\ln(bs)}{\ln(as)} \frac{as - 1}{bs - 1}}}, \quad x \in [0, 1], \quad s \in D, \quad (2.7)$$

where  $C_1$  and  $C_2$  are arbitrary functions which will be determined from the boundary conditions. Since the natural logarithm has the branch point at  $s = 0$ , we have  $D = \mathbb{C} \setminus (-\infty, 0]$  and this will be used in proposition 3.1.

Applying (2.5)<sub>1</sub> we obtain  $C = C_1 = -C_2$  and thus

$$\tilde{u}(x, s) = C(s) \left( e^{xs \sqrt{\frac{\ln(bs)}{\ln(as)} \frac{as-1}{bs-1}}} - e^{-xs \sqrt{\frac{\ln(bs)}{\ln(as)} \frac{as-1}{bs-1}}} \right), \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0].$$

Using (2.5)<sub>2</sub> in the previous expression, it follows

$$\tilde{u}(x, s) = \tilde{\Upsilon}(s) \tilde{P}(x, s), \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (2.8)$$

We introduced  $\tilde{P}$  as

$$\tilde{P}(x, s) = \frac{\sinh \left( xs \sqrt{\frac{\ln(bs)}{\ln(as)} \frac{as-1}{bs-1}} \right)}{\sinh \left( s \sqrt{\frac{\ln(bs)}{\ln(as)} \frac{as-1}{bs-1}} \right)}, \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (2.9)$$

Note that  $P(x, \cdot)$  is a distribution supported by  $[0, \infty)$ . It is clear that for  $x = 1$ ,  $P(1, t) = \delta(t)$ ,  $t \in \mathbb{R}$ . Actually, we will calculate  $P$  and show that for every  $x \in [0, 1]$ ,  $P(x, \cdot)$  is a locally integrable function on  $\mathbb{R}$ , equal to zero for  $t < 0$ .

Since  $\Upsilon$  and  $P$  are supported by  $[0, \infty)$ , displacement  $u$  is given by

$$\begin{aligned} u(x, t) &= \Upsilon(t) * P(x, t), \quad x \in [0, 1], \quad t \in \mathbb{R}, \quad \text{so that} \\ u(x, t) &= 0, \quad x \in [0, 1], \quad t < 0, \end{aligned} \quad (2.10)$$

where we use  $*$  to denote the convolution. Recall if  $f, g \in L^1_{loc}(\mathbb{R})$ ,  $\text{supp } f, g \subset [0, \infty)$ , then  $(f * g)(t) := \int_0^t f(\tau) g(t - \tau) d\tau$ ,  $t \in \mathbb{R}$ . Calculation of (2.10) will be done by the use of the Laplace inversion formula applied to (2.8). Moreover, we will show that  $u_H(x, t) = H(t) * P(x, t)$ ,  $t \in \mathbb{R}$ , is a continuous function, equal to zero for  $t < 0$ .

Next, we use (2.6)<sub>2</sub> and (2.6)<sub>3</sub> and obtain

$$\tilde{\sigma}(x, s) = \frac{\ln(as)}{\ln(bs)} \frac{bs-1}{as-1} \frac{\partial}{\partial x} \tilde{u}(x, s), \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (2.11)$$

In order to determine  $\sigma$ , we use (2.8) and (2.9) in (2.11), so that

$$\tilde{\sigma}(x, s) = s \tilde{\Upsilon}(s) \tilde{T}(x, s), \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0], \quad (2.12)$$

where

$$\tilde{T}(x, s) = \sqrt{\frac{\ln(as)}{\ln(bs)} \frac{bs-1}{as-1}} \frac{\cosh \left( xs \sqrt{\frac{\ln(bs)}{\ln(as)} \frac{as-1}{bs-1}} \right)}{\sinh \left( s \sqrt{\frac{\ln(bs)}{\ln(as)} \frac{as-1}{bs-1}} \right)}, \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (2.13)$$

Note that  $T$  is a locally integrable function for  $t > 0$ . For the determination of  $\sigma$ , we will use the Laplace inversion formula applied to (2.12) and obtain

$$\sigma(x, t) = \frac{d}{dt} (\Upsilon(t) * T(x, t)), \quad x \in [0, 1], \quad t \in \mathbb{R}, \quad (2.14)$$

where the derivative is understood in the sense of distributions. Again,  $\sigma(x, t) = 0$  for  $x \in [0, 1]$ ,  $t < 0$ . For the detailed account see §3.2.

### 3 Explicit forms of solutions

In this section we will calculate inverse Laplace transformations of distributions and functions on  $\mathbb{R}$  supported by  $[0, \infty)$ . For that purposes, let us define

$$M(s) := \sqrt{\frac{\ln(bs) as - 1}{\ln(as) bs - 1}}, \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (3.1)$$

In the sequel we will write  $A(x) \sim B(x)$  if  $\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1$ . Next proposition establishes some properties of  $M$ .

**Proposition 3.1.** (i)  $M$  is an analytic function in  $s \in \mathbb{C} \setminus (-\infty, 0]$ ;

$$(ii) \quad \lim_{\substack{s \rightarrow 0 \\ s \in \mathbb{C} \setminus (-\infty, 0]}} M(s) = 1 \text{ and } \lim_{\substack{|s| \rightarrow \infty \\ s \in \mathbb{C} \setminus (-\infty, 0]}} M(s) = \sqrt{\frac{a}{b}}.$$

(iii) Let  $p \in (0, s_0)$ ,  $s_0 > 0$ . Then

$$M(p \pm iR) \sim \sqrt{\frac{a}{b}} \frac{1}{\ln(aR)} \sqrt{(\ln(aR) \ln(bR))^2 + \left(\frac{\pi}{2} \ln \frac{b}{a}\right)^2} e^{\mp i \arctan \frac{\frac{\pi}{2} \ln \frac{b}{a}}{\ln(aR) \ln(bR)}}$$

as  $R \rightarrow \infty$ .

*Proof.* We first prove (i). The only points where  $M$  could be singular are  $s = \frac{1}{a}$  and  $s = \frac{1}{b}$ . Since,

$$\frac{\ln(bs)}{bs-1} = \frac{\ln(1+(bs-1))}{bs-1} = \sum_{n=0}^{\infty} \frac{(-1)^n (bs-1)^n}{n+1}, \quad |bs-1| \in (-1, 1],$$

it is obvious that  $s = \frac{1}{b}$  is a regular point of  $M$ . Similar arguments hold for  $s = \frac{1}{a}$ . Limits in (ii) can easily be calculated. In order to prove (iii), let us introduce  $\mu = \sqrt{p^2 + R^2}$  and  $\nu = \arctan \frac{R}{p}$ . It is obvious that  $\mu \sim R$ ,  $\nu \sim \frac{\pi}{2}$ , as  $R \rightarrow \infty$ . Then  $M(p \pm iR)$  becomes

$$\begin{aligned} & M(p \pm iR) \\ &= \sqrt{\frac{\ln(a\mu) \ln(b\mu) + \nu^2 \mp i\nu \ln \frac{b}{a}}{\ln^2(a\mu) + \nu^2}} \sqrt{\frac{(ap-1)(bp-1) + abR^2 \pm iR(b-a)}{(bp-1)^2 + (bR)^2}} \\ &= \left( \frac{[abR^2 + (ap-1)(bp-1)] [\ln(a\mu) \ln(b\mu) + \nu^2] + R\nu(b-a) \ln \frac{b}{a}}{((bR)^2 + (bp-1)^2) (\ln^2(a\mu) + \nu^2)} \right. \\ & \quad \left. \pm i \frac{R(b-a) [\ln(a\mu) \ln(b\mu) + \nu^2] - \nu \ln \frac{b}{a} [abR^2 + (ap-1)(bp-1)]}{((bR)^2 + (bp-1)^2) (\ln^2(a\mu) + \nu^2)} \right)^{\frac{1}{2}} \\ &\sim \sqrt{\frac{a}{b}} \frac{1}{\ln(aR)} \sqrt{\ln(aR) \ln(bR) \mp i \frac{\pi}{2} \ln \frac{b}{a}}, \quad \text{as } R \rightarrow \infty. \end{aligned}$$

□

### 3.1 Determination of displacement $u$ in case of stress relaxation

We investigate properties of  $\tilde{P}$ , given by (2.9), and find a solution to (2.3), (2.4), (2.5), with weight functions given by (1.5), in two steps. First, we find a solution with the boundary condition (2.5) in the case when

$$\Upsilon(t) = \Upsilon_0 H(t), \quad \Upsilon_0 > 0, \quad t \in \mathbb{R}. \quad (3.2)$$

Then we proceed to a more general case, when  $\Upsilon$  is assumed to be of the form

$$\Upsilon(t) = \Upsilon_0 H(t) + F(t), \quad t \in \mathbb{R}, \quad (3.3)$$

where  $F$  is a locally integrable function, equal to zero on  $(-\infty, 0]$ . Additional assumptions on  $F$  will be postulated in §3.1.2.

Let us examine properties of  $\tilde{P}$ , given by (2.9). Clearly, it has complex conjugated poles at  $ps_n^{(\pm)}$ ,  $n \in \mathbb{N}$ . Poles are solutions of

$$\sinh(sM(s)) = 0, \quad \text{i.e. } sM(s) = \pm ni\pi. \quad (3.4)$$

Let us examine the position and the multiplicity of solutions to (3.4).

**Proposition 3.2.** *There are infinitely many solutions  $ps_n^{(\pm)}$ ,  $n \in \mathbb{N}$ , of (3.4), such that*

$$\operatorname{Re}\left(ps_n^{(\pm)}\right) \approx -\frac{\frac{\pi}{4} \ln \frac{b}{a} \sqrt{\frac{b}{a}} n\pi}{\ln(\sqrt{ab}n\pi) \ln\left(b\sqrt{\frac{b}{a}}n\pi\right)}, \quad (3.5)$$

$$\operatorname{Im}\left(ps_n^{(\pm)}\right) \approx \pm R \approx \pm \sqrt{\frac{b}{a}} n\pi, \quad (3.6)$$

as  $n \rightarrow \infty$ . Moreover, there exists  $n_0 \in \mathbb{N}$ , such that poles  $ps_n^{(\pm)}$ , for  $n > n_0$ , are simple.

*Proof.* Let us square (3.4) and put  $ps_n^{(\pm)} = Re^{i\phi}$ ,  $\phi \in (-\pi, \pi)$ . Then, after separation of real and imaginary parts, we obtain

$$R^2 \cos(2\phi) \operatorname{Re}(M^2(Re^{i\phi})) - R^2 \sin(2\phi) \operatorname{Im}(M^2(Re^{i\phi})) = -n^2\pi^2, \quad (3.7)$$

$$R^2 \sin(2\phi) \operatorname{Re}(M^2(Re^{i\phi})) + R^2 \cos(2\phi) \operatorname{Im}(M^2(Re^{i\phi})) = 0. \quad (3.8)$$

By the use of (3.1), real and imaginary parts of  $M^2(Re^{i\phi})$  are

$$\begin{aligned} \operatorname{Re}(M^2(Re^{i\phi})) &= \frac{(\ln(aR) \ln(bR) + \phi^2)(abR^2 - (a+b)R \cos \phi + 1)}{(\ln^2(aR) + \phi^2)(b^2R^2 - 2bR \cos \phi + 1)} \\ &\quad + \frac{\ln \frac{b}{a} (b-a) R \phi \sin \phi}{(\ln^2(aR) + \phi^2)(b^2R^2 - 2bR \cos \phi + 1)}, \\ \operatorname{Im}(M^2(Re^{i\phi})) &= -\frac{\phi \ln \frac{b}{a} (abR^2 - (a+b)R \cos \phi + 1)}{(\ln^2(aR) + \phi^2)(b^2R^2 - 2bR \cos \phi + 1)} \\ &\quad + \frac{(b-a) R \sin \phi (\ln(aR) \ln(bR) + \phi^2)}{(\ln^2(aR) + \phi^2)(b^2R^2 - 2bR \cos \phi + 1)}. \end{aligned}$$

Letting  $R \rightarrow \infty$ , previous expressions are written as

$$\operatorname{Re} (M^2 (Re^{i\phi})) \approx \frac{abR^2 \ln(aR) \ln(bR)}{b^2 R^2 \ln^2(aR)} = \frac{a \ln(bR)}{b \ln(aR)}, \quad (3.9)$$

$$\operatorname{Im} (M^2 (Re^{i\phi})) \approx -\frac{ab \ln \frac{b}{a} R^2 \phi}{b^2 R^2 \ln^2(aR)} = -\frac{a}{b} \ln \frac{b}{a} \phi \frac{1}{\ln^2(aR)}. \quad (3.10)$$

Using (3.8), (3.9) and (3.10), we obtain

$$\tan(2\phi) = -\frac{\operatorname{Im} (M^2 (Re^{i\phi}))}{\operatorname{Re} (M^2 (Re^{i\phi}))} \approx \phi \frac{\ln \frac{b}{a}}{\ln(aR) \ln(bR)}. \quad (3.11)$$

Let  $\phi \in (0, \pi)$ . Then  $\frac{\tan(2\phi)}{\phi} > 0$  and  $\frac{\tan(2\phi)}{\phi} \rightarrow 0$  as  $R \rightarrow \infty$ . Hence,  $\phi \in (0, \frac{\pi}{4})$  or  $\phi \in (\frac{\pi}{2}, \frac{3\pi}{4})$ . Since  $\phi \neq 0$  and  $\tan(2\phi) \rightarrow 0$ , it follows that  $\phi \rightarrow \frac{\pi}{2}$  from the interval  $\phi \in (\frac{\pi}{2}, \frac{3\pi}{4})$ . Therefore, by (3.11), we have

$$\sin \phi \approx 1 - \left( \frac{\frac{\pi}{2} \ln \frac{b}{a}}{2 \ln(aR) \ln(bR)} \right)^2 \approx 1, \quad \cos \phi \approx -\frac{\frac{\pi}{2} \ln \frac{b}{a}}{2 \ln(aR) \ln(bR)}. \quad (3.12)$$

Inserting (3.9), (3.10) and (3.12) in (3.7), we obtain

$$\frac{1}{\sqrt{\ln^2(aR) \ln^2(bR) + \left(\frac{\pi}{2} \ln \frac{b}{a}\right)^2}} \left( \ln^2(bR) + \frac{\left(\frac{\pi}{2} \ln \frac{b}{a}\right)^2}{\ln^2(aR)} \right) \approx \frac{b}{a} \frac{n^2 \pi^2}{R^2} \approx 1. \quad (3.13)$$

Thus, real and imaginary parts of  $ps_n^{(\pm)}$ , as  $R \rightarrow \infty$ , obtained by (3.12) and (3.13), are as stated in proposition.

In order to prove that solutions to (3.4) are simple for  $n > n_0$ , we define

$$f(s) := \sinh(sM(s)), \quad s \in \mathbb{C} \setminus (-\infty, 0].$$

Then  $(s \in \mathbb{C} \setminus (-\infty, 0])$

$$\frac{d}{ds} f(s) = M(s) \left( 1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{(b-a)s}{2(as-1)(bs-1)} \right) \cosh(sM(s)).$$

Solutions to  $f(s) = 0$  are given by (3.4), and so, as  $|ps_n^{(\pm)}| \rightarrow \infty$ ,

$$\frac{d}{ds} f(s) \Big|_{s=ps_n^{(\pm)}} \sim (-1)^n \left[ M(s) \left( 1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{(b-a)s}{2(as-1)(bs-1)} \right) \right]_{s=ps_n^{(\pm)}}.$$

By proposition 3.1,  $M \sim \sqrt{\frac{a}{b}}$  and this implies that

$$\frac{d}{ds} f(s) \Big|_{s=ps_n^{(\pm)}} \sim (-1)^n \sqrt{\frac{a}{b}} \quad \text{as} \quad |ps_n^{(\pm)}| \rightarrow \infty.$$

Thus, for large  $|ps_n^{(\pm)}|$  we have  $\frac{d}{ds} f(s) \Big|_{s=ps_n^{(\pm)}} \neq 0$  and solutions are simple for  $n > n_0$ .  $\square$



### 3.1.1 Case $\Upsilon = \Upsilon_0 H$

This is the case that has physical importance, since we obtain displacement in case of stress relaxation test. Formally, we write (2.10) with (3.2) as

$$u_H(x, t) = \Upsilon_0 H(t) * P(x, t), \quad x \in [0, 1], \quad t \in \mathbb{R}. \quad (3.14)$$

The following theorem is on existence and properties of  $u_H$ .

**Theorem 3.3.** *Let  $\Upsilon = \Upsilon_0 H$  and let  $\phi_1$  and  $\phi_2$  be given by (1.5). Then the solution to (2.3), (2.4), (2.5) is given by (3.14), where*

$$\begin{aligned} P(x, t) &= \frac{1}{2\pi i} \int_0^\infty \left( \frac{\sinh(xqM(qe^{-i\pi}))}{\sinh(qM(qe^{-i\pi}))} - \frac{\sinh(xqM(qe^{i\pi}))}{\sinh(qM(qe^{i\pi}))} \right) e^{-qt} dq \\ &\quad + \sum_{n=1}^\infty \left[ \text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(+)} \right) \right. \\ &\quad \left. + \text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(-)} \right) \right], \quad x \in [0, 1], \quad t > 0, \end{aligned} \quad (3.15)$$

$$P(x, t) = 0, \quad x \in [0, 1], \quad t < 0. \quad (3.16)$$

The residues are given by

$$\text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(\pm)} \right) = \left[ \frac{\sinh(xsM(s))}{\frac{d}{ds} [\sinh(sM(s))]} e^{st} \right]_{s={}_P s_n^{(\pm)}} \quad (3.17)$$

and simple poles  ${}_P s_n^{(\pm)}$ , for  $n > n_0$ , are solutions of (3.4). Function  $P$  is real-valued, locally integrable on  $\mathbb{R}$  and smooth for  $t > 0$ .

The explicit form of solution is

$$\begin{aligned} u_H(x, t) &= \frac{\Upsilon_0}{2\pi i} \int_0^\infty \left( \frac{\sinh(xqM(qe^{-i\pi}))}{\sinh(qM(qe^{-i\pi}))} - \frac{\sinh(xqM(qe^{i\pi}))}{\sinh(qM(qe^{i\pi}))} \right) \frac{1 - e^{-qt}}{q} dq \\ &\quad + \int_0^t \left( \sum_{n=1}^\infty \left[ \text{Res} \left( \tilde{P}(x, s) e^{s\tau}, {}_P s_n^{(+)} \right) \right. \right. \\ &\quad \left. \left. + \text{Res} \left( \tilde{P}(x, s) e^{s\tau}, {}_P s_n^{(-)} \right) \right] \right) d\tau, \quad x \in [0, 1], \quad t > 0, \end{aligned} \quad (3.18)$$

$$u_H(x, t) = 0, \quad x \in [0, 1], \quad t < 0. \quad (3.19)$$

Function  $u_H$  is continuous at  $t = 0$ .

*Proof.* We calculate  $P(x, t)$ ,  $x \in [0, 1]$ ,  $t \in \mathbb{R}$ , by the integration over a suitable contour.

Let  $t > 0$ . The Cauchy residues theorem yields

$$\oint_\Gamma \tilde{P}(x, s) e^{st} ds = 2\pi i \sum_{n=1}^\infty \left[ \text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(+)} \right) + \text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(-)} \right) \right], \quad (3.20)$$

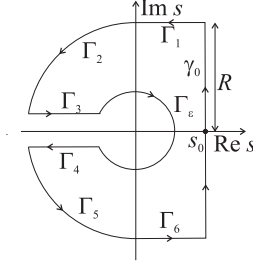


Figure 3.1: Integration contour  $\Gamma$

where  $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_\varepsilon \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \gamma_0$ , so that all poles lie inside the contour  $\Gamma$  (see figure 3.1).

First we show that the series of residues in (3.15) is convergent. By proposition 3.2, poles  ${}_P s_n^{(\pm)}$  of  $\tilde{P}$ , given by (2.9), are simple for  $n > n_0$ . Then residues in (3.20) can be calculated as it is given in (3.17). We use (3.4) to write (3.17) as

$$\begin{aligned} & \text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(\pm)} \right) \\ &= (-1)^n \frac{\sin(n\pi x)}{n\pi} \left[ \frac{se^{st}}{1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)}} \right]_{s={}_P s_n^{(\pm)}}, \quad n > n_0. \end{aligned} \quad (3.21)$$

Let  ${}_P s_n^{(\pm)} = Re^{\pm i\phi}$ , then (3.21) transforms into

$$\begin{aligned} & \text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(\pm)} \right) \\ &= (-1)^n \frac{\sin(n\pi x)}{n\pi} \frac{Re^{Rt \cos \phi} e^{\pm i(\phi + Rt \sin \phi)}}{\left[ 1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)} \right]_{s=Re^{\pm i\phi}}} \\ &= (-1)^n \frac{\sin(n\pi x)}{n\pi} \frac{Re^{Rt \cos \phi} [\cos(\phi + Rt \sin \phi) \pm i \sin(\phi + Rt \sin \phi)]}{\left[ 1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)} \right]_{s=Re^{\pm i\phi}}}, \end{aligned}$$

and therefore, for  $n > n_0$ , we have

$$\begin{aligned} & \text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(+)} \right) + \text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(-)} \right) = \\ & (-1)^n \frac{\sin(n\pi x)}{n\pi} Re^{Rt \cos \phi} \\ & \times \left( \frac{\cos(\phi + Rt \sin \phi) + i \sin(\phi + Rt \sin \phi)}{\left[ 1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)} \right]_{s=Re^{i\phi}}} \right. \\ & \left. + \frac{\cos(\phi + Rt \sin \phi) - i \sin(\phi + Rt \sin \phi)}{\left[ 1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)} \right]_{s=Re^{-i\phi}}} \right). \end{aligned} \quad (3.22)$$

Let  $n \rightarrow \infty$  (then also  $|_P s_n^{(\pm)}| \rightarrow \infty$ , i.e.  $R \rightarrow \infty$ ). Then

$$\left| \left[ 1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)} \right]_{s=Re^{\pm i\phi}} \right| \rightarrow 1,$$

and (3.22), as  $n \rightarrow \infty$ , becomes

$$\begin{aligned} & \left| \text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(+)} \right) + \text{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(-)} \right) \right| \\ & \approx 2 \left| \frac{\sin(n\pi x)}{n\pi} R e^{Rt \cos \phi} \cos(\phi + Rt \sin \phi) \right|. \end{aligned}$$

Proposition 3.2, (3.5), implies that

$$\text{Re} \left( {}_P s_n^{(\pm)} \right) \approx -\frac{\pi}{4} \ln \frac{b}{a} \sqrt{\frac{b}{a}} \pi \frac{n}{\ln(\sqrt{ab}n\pi) \ln\left(b\sqrt{\frac{b}{a}}n\pi\right)} \leq -C\sqrt{n}, \quad n > n_0.$$

Also, by (3.6), we have that  $\frac{R}{n} \approx \sqrt{\frac{b}{a}}\pi$ . This implies that summands in (3.20) can be estimated by  $K e^{-Ct\sqrt{n}}$ , which implies the convergence of the sum of residues in (3.20).

Second, we calculate the integral over  $\Gamma$  in (3.20). Consider the integral along contour  $\Gamma_1$ . Then

$$\left| \int_{\Gamma_1} \tilde{P}(x, s) e^{st} ds \right| \leq \int_0^{s_0} \left| \tilde{P}(x, p + iR) \right| \left| e^{(p+iR)t} \right| dp.$$

Let  $R \rightarrow \infty$ . In order to estimate  $\left| \tilde{P}(x, p \pm iR) \right|$ , using (iii) of proposition 3.1, we write

$$\begin{aligned} M(p \pm iR) & \sim v \pm iw, \\ v & = \sqrt{\frac{a}{b}} \frac{1}{\ln(aR)} \frac{\ln(aR) \ln(bR)}{\sqrt[4]{(\ln(aR) \ln(bR))^2 + \left(\frac{\pi}{2} \ln \frac{b}{a}\right)^2}}, \\ w & = -\sqrt{\frac{a}{b}} \frac{1}{\ln(aR)} \frac{\frac{\pi}{2} \ln \frac{b}{a}}{\sqrt[4]{(\ln(aR) \ln(bR))^2 + \left(\frac{\pi}{2} \ln \frac{b}{a}\right)^2}}. \end{aligned}$$

Then, as  $R \rightarrow \infty$ ,

$$\begin{aligned} \left| \tilde{P}(x, p \pm iR) \right| & \sim \left| \frac{\sinh[x(pv - Rw) \pm ix(pw + Rv)]}{\sinh[(pv - Rw) \pm i(pw + Rv)]} \right| \\ & \leq \frac{e^{x(pv - Rw)} + e^{-x(pv - Rw)}}{\left| e^{pv - Rw} - e^{-(pv - Rw)} \right|} \\ & = e^{-(1-x)(pv - Rw)} \frac{1 + e^{-2x(pv - Rw)}}{\left| 1 - e^{-2(pv - Rw)} \right|} \rightarrow 0. \end{aligned} \tag{3.23}$$

The previous statement is valid since, as  $R \rightarrow \infty$ ,

$$\begin{aligned} pv - Rw &= \sqrt{\frac{a}{b}} \frac{1}{\ln(aR)} \frac{1}{\sqrt[4]{(\ln(aR) \ln(bR))^2 + \left(\frac{\pi}{2} \ln \frac{b}{a}\right)^2}} \\ &\times \left( p \ln(aR) \ln(bR) + R \frac{\pi}{2} \ln \frac{b}{a} \right) \\ &\sim \sqrt{\frac{a}{b}} \left( p \sqrt{\frac{\ln(bR)}{\ln(aR)}} + \frac{\pi}{2} \ln \frac{b}{a} \frac{R}{\ln(aR) \sqrt{\ln(aR) \ln(bR)}} \right) \rightarrow \infty. \end{aligned}$$

Therefore, according to (3.23), we have

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma_1} \tilde{P}(x, s) e^{st} ds \right| = 0.$$

By the use of (3.23), we conclude that similar arguments are valid for the integral along the contour  $\Gamma_6$ . Thus,

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma_6} \tilde{P}(x, s) e^{st} ds \right| = 0.$$

Next, we consider the integral along contour  $\Gamma_2$

$$\left| \int_{\Gamma_2} \tilde{P}(x, s) e^{st} ds \right| \leq \int_{\frac{\pi}{2}}^{\pi} R \left| e^{R(1-x)e^{i\phi}M(Re^{i\phi})} \right| \left| \frac{e^{2xRe^{i\phi}M(Re^{i\phi})} - 1}{e^{2Re^{i\phi}M(Re^{i\phi})} - 1} \right| e^{Rt \cos \phi} d\phi.$$

Since  $M \sim \sqrt{\frac{a}{b}}$  as  $|s| \rightarrow \infty$  and  $\cos \phi \leq 0$  for  $\phi \in [\frac{\pi}{2}, \pi]$ , by the Lebesgue theorem, we have

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma_2} \tilde{P}(x, s) e^{st} ds \right| \leq \lim_{R \rightarrow \infty} \int_{\frac{\pi}{2}}^{\pi} R e^{R \cos \phi (t + (1-x)\sqrt{\frac{a}{b}})} d\phi = 0.$$

Similar arguments are valid for the integral along the contour  $\Gamma_5$ . Thus,

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma_5} \tilde{P}(x, s) e^{st} ds \right| = 0.$$

The integration along contour  $\Gamma_\varepsilon$  gives

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left| \int_{\Gamma_\varepsilon} \tilde{P}(x, s) e^{st} ds \right| &= \lim_{\varepsilon \rightarrow 0} \int_{\pi}^{-\pi} \varepsilon \left| e^{-\varepsilon(1-x)e^{i\phi}M(\varepsilon e^{i\phi})} \right| \\ &\times \left| \frac{1 - e^{-2x\varepsilon e^{i\phi}M(\varepsilon e^{i\phi})}}{1 - e^{-2\varepsilon e^{i\phi}M(\varepsilon e^{i\phi})}} \right| e^{\varepsilon t \cos \phi} d\phi = 0. \end{aligned}$$

Integrals along parts of contour  $\Gamma_3$ ,  $\Gamma_4$  and  $\gamma_0$  give

$$\begin{aligned} \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{\Gamma_3} \tilde{P}(x, s) e^{st} ds &= \int_0^\infty \frac{\sinh(xqM(qe^{i\pi}))}{\sinh(qM(qe^{i\pi}))} e^{-qt} dq, \\ \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{\Gamma_4} \tilde{P}(x, s) e^{st} ds &= - \int_0^\infty \frac{\sinh(xqM(qe^{-i\pi}))}{\sinh(qM(qe^{-i\pi}))} e^{-qt} dq, \\ \lim_{R \rightarrow \infty} \int_{\gamma_0} \tilde{P}(x, s) e^{st} ds &= 2\pi i P(x, t). \end{aligned}$$

Now, by the Cauchy residues theorem (3.20), the function  $P$  is determined by (3.15).

In order to see that  $P$  is a real-valued function, we use  $M(qe^{\pm i\pi})$ ,  $q \in [0, \infty)$ , and note that  $M(qe^{-i\pi}) = \overline{M(qe^{i\pi})}$ , where the bar denotes the complex conjugation. Due to the exponential in the hyperbolic sine, we have  $\sinh(xqM(qe^{-i\pi})) = \overline{\sinh(xqM(qe^{i\pi}))}$  and therefore the integrand in (3.15) is of the form

$$\begin{aligned} & \frac{\sinh(xqM(qe^{-i\pi}))}{\sinh(qM(qe^{-i\pi}))} - \frac{\sinh(xqM(qe^{i\pi}))}{\sinh(qM(qe^{i\pi}))} \\ &= \overline{\left( \frac{\sinh(xqM(qe^{i\pi}))}{\sinh(qM(qe^{i\pi}))} \right)} - \frac{\sinh(xqM(qe^{i\pi}))}{\sinh(qM(qe^{i\pi}))} \\ &= -2i \operatorname{Im} \left( \frac{\sinh(xqM(qe^{i\pi}))}{\sinh(qM(qe^{i\pi}))} \right), \end{aligned}$$

which implies that the first term in (3.15) is real.

Next, we examine  $\operatorname{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(\pm)} \right)$  in order to prove that the sum of residues is also real. By (3.21) and

$$\begin{aligned} & \left[ 1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{(b-a)s}{2(as-1)(bs-1)} \right]_{s={}_P s_n^{(-)}} \\ &= \overline{\left( \left[ 1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{(b-a)s}{2(as-1)(bs-1)} \right]_{s={}_P s_n^{(+)}} \right)} \end{aligned}$$

we obtain that  $\operatorname{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(-)} \right) = \overline{\operatorname{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(+)} \right)}$ . It is clear that

$$\operatorname{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(+)} \right) + \operatorname{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(-)} \right) = 2 \operatorname{Re} \left( \operatorname{Res} \left( \tilde{P}(x, s) e^{st}, {}_P s_n^{(+)} \right) \right).$$

This implies that the second term in (3.15) is also real for  $n \in \mathbb{N}$ . Hence, (3.15) is a real-valued function.

Let  $t < 0$ . We prove that the integral over  $\gamma_0$  does not depend on the choice of  $s_0$  (see figure 3.1). Let  $\bar{\Gamma} = \gamma_0 \cup \gamma_1 \cup \gamma'_0 \cup \gamma_2$  (see figure 3.2), where  $s_0$  and  $s'_0$  are chosen so that all poles, i.e. solutions of (3.4), lie on the left of  $\gamma_0$ . The Cauchy residues theorem yields ( $x \in [0, 1]$ )

$$\oint_{\bar{\Gamma}} \tilde{P}(x, s) e^{st} ds = 0.$$

This and (3.23) imply

$$\lim_{R \rightarrow \infty} \left| \int_{\gamma_1} \tilde{P}(x, s) e^{st} ds \right| \leq \lim_{R \rightarrow \infty} \int_{s_0}^{s'_0} \left| \tilde{P}(x, v + iR) \right| \left| e^{(v+iR)t} \right| dv = 0.$$

Similar arguments hold for the integral along  $\gamma_2$ . Therefore, by the Cauchy residues theorem, integrals along  $\gamma_0$  and  $\gamma'_0$  are equal and the inversion of the Laplace transformation does not depend on the choice of  $s_0$  as well as on the choice of  $s'_0$ .

The Cauchy residues theorem yields ( $x \in [0, 1]$ )

$$\oint_{\tilde{\Gamma}} \tilde{P}(x, s) e^{st} ds = 0,$$

where  $\tilde{\Gamma} = \gamma_0 \cup \Gamma_r$  (see figure 3.3), with the assumption that all poles, i.e. solutions of (3.4), lie on the left of  $\gamma_0$ . Let  $r(R) = \sqrt{R^2 + s_0^2}$ . Consider

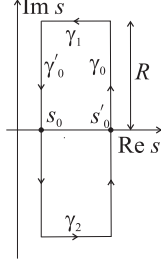


Figure 3.2: Integration contour  $\tilde{\Gamma}$

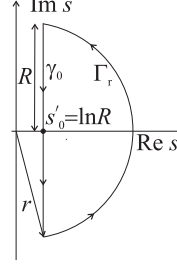


Figure 3.3: Integration contour  $\tilde{\Gamma}$

$$\left| \int_{\Gamma_r} \tilde{P}(x, s) e^{st} ds \right| \leq \int_{-\phi_0(r(R))}^{\phi_0(r(R))} \sqrt{R^2 + s_0^2} \left| \tilde{P}\left(x, \sqrt{R^2 + s_0^2} e^{i\phi}\right) \right| e^{t\sqrt{R^2 + s_0^2} \cos \phi} d\phi,$$

where  $\lim_{R \rightarrow \infty} \phi_0(r(R)) = \frac{\pi}{2}$ . Since  $M \sim \sqrt{\frac{a}{b}}$  as  $|s| \rightarrow \infty$ , by (2.9) and (3.1), we have, as  $|s| \rightarrow \infty$ ,

$$\left| \tilde{P}(x, s) \right| = \left| e^{-(1-x)sM(s)} \frac{1 - e^{-2xsM(s)}}{1 - e^{-2sM(s)}} \right| \leq C, \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (3.24)$$

By (3.24), we have

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma_r} \tilde{P}(x, s) e^{st} ds \right| \leq C \lim_{R \rightarrow \infty} \int_{-\phi_0(r(R))}^{\phi_0(r(R))} \sqrt{R^2 + s_0^2} e^{t\sqrt{R^2 + s_0^2} \cos \phi} d\phi = 0,$$

since  $t < 0$  and  $\cos \phi > 0$ . Therefore, we proved (3.16).

By the use of (3.15) and (3.16) in (3.14) and by calculating the convolution we obtain (3.18) and (3.19).

In order to prove that  $u_H$  is a continuous function at  $t = 0$ , we will use Lebesgue dominated convergence theorem. Let

$$f(q, x) := \frac{\Upsilon_0}{2\pi i} \left( \frac{\sinh(xqM(qe^{-i\pi}))}{\sinh(qM(qe^{-i\pi}))} - \frac{\sinh(xqM(qe^{i\pi}))}{\sinh(qM(qe^{i\pi}))} \right), \quad q \in (0, \infty), \quad x \in [0, 1].$$

Then

$$\left| \int_0^\infty f(q, x) \frac{1 - e^{-qt}}{q} dq \right| \rightarrow 0 \quad \text{as } t \rightarrow 0. \quad (3.25)$$

By simple calculations we have  $\frac{1 - e^{-qt}}{q} \leq Ct$  if  $0 < q < 1$  and  $\frac{1 - e^{-qt}}{q} \leq 1 - e^{-qt}$  if  $q \geq 1$ . Thus

$$f(q, x) \frac{1 - e^{-qt}}{q} \leq Cf(q, x), \quad q > 0$$

and since  $\frac{1-e^{-qt}}{q} \rightarrow 0$  as  $t \rightarrow 0$ , (3.25) follows.

In proving the continuity of  $u_H$  at  $t = 0$ , by (3.14), we estimated

$$\int_0^t P(x, \tau) d\tau, \quad x \in [0, 1], \quad t > 0,$$

and actually proved that  $P$  is integrable function on any interval  $[0, T]$ ,  $T > 0$ . Thus,  $P$  is locally integrable on  $\mathbb{R}$ .  $\square$

### 3.1.2 Case $\Upsilon = \Upsilon_0 H + F$

**Condition 3.4.** Let  $F$  be a locally integrable function, equal to zero for  $t \leq 0$ , such that its Laplace transformation exists in  $\mathbb{C} \setminus (-\infty, 0]$ . Assume:

- (i)  $\tilde{F}$  is analytic and  $\tilde{F} \neq 0$  in  $\mathbb{C} \setminus (-\infty, 0]$ ;
- (ii) for some  $\alpha > 1$ ,  $\tilde{F}(s) \sim \frac{1}{|s|^\alpha}$ ,  $s \in \mathbb{C} \setminus (-\infty, 0]$ , as  $|s| \rightarrow \infty$ ;
- (iii)  $s\tilde{F}(s) \sim o(1)$ ,  $s \in \mathbb{C} \setminus (-\infty, 0]$ , as  $|s| \rightarrow 0$ .

If the boundary condition (2.5) is given by (3.3), then the solution to (2.3), (2.4), (2.5), given by (2.8) in the Laplace domain, reads formally

$$\tilde{u}(x, s) = \tilde{u}_H(x, s) + \tilde{F}(s) \tilde{P}(x, s), \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0],$$

and in the time domain it is

$$u(x, t) = u_H(x, t) + F(t) * P(x, t), \quad x \in [0, 1], \quad t \in \mathbb{R}.$$

The existence of  $u_H$  is shown in §3.1.1, therefore it remains to show the existence of

$$u_F(x, t) = F(t) * P(x, t), \quad x \in [0, 1], \quad t \in \mathbb{R}.$$

Let  $t > 0$ . The Cauchy residues theorem yields ( $x \in [0, 1]$ )

$$\oint_{\Gamma} \tilde{u}_F(x, s) e^{st} ds = 2\pi i \sum_{n=1}^{\infty} \left[ \text{Res} \left( \tilde{u}_F(x, s) e^{st}, {}_P s_n^{(+)} \right) + \text{Res} \left( \tilde{u}_F(x, s) e^{st}, {}_P s_n^{(-)} \right) \right], \quad (3.26)$$

where  $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_\varepsilon \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \gamma_0$  (see figure 3.1). Since poles  ${}_P s_n^{(\pm)}$  of  $\tilde{u}_F$  are actually the poles of  $\tilde{P}$ , that are obtained from (3.4) and they are simple for  $n > n_0$ , the residues in (3.26) can be calculated as

$$\text{Res} \left( \tilde{u}_F(x, s) e^{st}, {}_P s_n^{(\pm)} \right) = \left[ \tilde{F}(s) \frac{\sinh(xsM(s))}{\frac{d}{ds} [\sinh(sM(s))]} e^{st} \right]_{s={}_P s_n^{(\pm)}}. \quad (3.27)$$

The proof that the sum in (3.26) converges is analog to the one presented in §3.1.1.

Consider the integral along contour  $\Gamma_1$ . It reads

$$\left| \int_{\Gamma_1} \tilde{u}_F(x, s) e^{st} ds \right| \leq \int_0^{s_0} \left| \tilde{F}(p + iR) \right| \left| \tilde{P}(x, p + iR) \right| \left| e^{(p+iR)t} \right| dp.$$

According to (3.24) and condition 3.4, we have

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma_1} \tilde{u}_F(x, s) e^{st} ds \right| \leq C \lim_{R \rightarrow \infty} \int_0^{s_0} \frac{1}{\left( \sqrt{p^2 + R^2} \right)^\alpha} e^{pt} dp = 0.$$

The integral along contour  $\Gamma_2$  reads

$$\begin{aligned} \left| \int_{\Gamma_2} \tilde{u}_F(x, s) e^{st} ds \right| &\leq \int_{\frac{\pi}{2}}^{\pi} \left| \tilde{F}(Re^{i\phi}) \right| \left| e^{R(1-x)e^{i\phi}M(Re^{i\phi})} \right| \\ &\quad \times \left| \frac{e^{2xRe^{i\phi}M(Re^{i\phi})} - 1}{e^{2Re^{i\phi}M(Re^{i\phi})} - 1} \right| e^{Rt \cos \phi} R d\phi. \end{aligned}$$

In order to apply the Lebesgue theorem, we need  $\alpha > 1$  in condition 3.4 (actually it is enough to have  $\alpha \geq 1$ , but case  $\alpha = 1$  is already considered). Since  $M \sim \sqrt{\frac{2}{b}}$  as  $|s| \rightarrow \infty$  and  $\cos \phi \leq 0$  for  $\phi \in [\frac{\pi}{2}, \pi]$ , we have

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma_2} \tilde{u}_F(x, s) e^{st} ds \right| \leq C \lim_{R \rightarrow \infty} \int_{\frac{\pi}{2}}^{\pi} R^{1-\alpha} e^{R \cos \phi (t + (1-x)\sqrt{\frac{2}{b}})} d\phi = 0.$$

Similar arguments are valid for the integral along the contour  $\Gamma_5$ . Thus,

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma_5} \tilde{u}_F(x, s) e^{st} ds \right| = 0.$$

The integration along contour  $\Gamma_\varepsilon$  gives

$$\begin{aligned} \left| \int_{\Gamma_\varepsilon} \tilde{u}_F(x, s) e^{st} ds \right| &\leq \int_{-\pi}^{\pi} \left| \tilde{F}(\varepsilon e^{i\phi}) \right| \left| e^{-\varepsilon(1-x)e^{i\phi}M(\varepsilon e^{i\phi})} \right| \\ &\quad \times \left| \frac{1 - e^{-2x\varepsilon e^{i\phi}M(\varepsilon e^{i\phi})}}{1 - e^{-2\varepsilon e^{i\phi}M(\varepsilon e^{i\phi})}} \right| e^{\varepsilon t \cos \phi} \varepsilon d\phi, \end{aligned}$$

and this tends to zero as  $\varepsilon \rightarrow 0$ , according to condition 3.4. Integrals along parts of contour  $\Gamma_3$ ,  $\Gamma_4$  and  $\gamma_0$  give

$$\begin{aligned} \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{\Gamma_3} \tilde{u}_F(x, s) e^{st} ds &= \int_0^\infty \tilde{F}(qe^{i\pi}) \frac{\sinh(xqM(qe^{i\pi}))}{\sinh(qM(qe^{i\pi}))} e^{-qt} dq, \\ \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{\Gamma_4} \tilde{u}_F(x, s) e^{st} ds &= - \int_0^\infty \tilde{F}(qe^{-i\pi}) \frac{\sinh(xqM(qe^{-i\pi}))}{\sinh(qM(qe^{-i\pi}))} e^{-qt} dq, \\ \lim_{R \rightarrow \infty} \int_{\gamma_0} \tilde{u}_F(x, s) e^{st} ds &= 2\pi i u_F(x, t). \end{aligned}$$



Now, by the Cauchy residues theorem (3.26),  $u_F$  is determined as

$$\begin{aligned}
u_F(x, t) &= \frac{1}{2\pi i} \int_0^\infty \left( \tilde{F}(qe^{-i\pi}) \frac{\sinh(xqM(qe^{-i\pi}))}{\sinh(qM(qe^{-i\pi}))} \right. \\
&\quad \left. - \tilde{F}(qe^{i\pi}) \frac{\sinh(xM(qe^{i\pi}))}{\sinh(qM(qe^{i\pi}))} \right) e^{-qt} dq \\
&\quad + \sum_{n=1}^\infty \left[ \text{Res} \left( \tilde{u}_F e^{st}, {}_P s_n^{(+)} \right) + \text{Res} \left( \tilde{u}_F e^{st}, {}_P s_n^{(-)} \right) \right], \quad x \in [0, 1], \quad t > 0, \\
u_F(x, t) &= 0, \quad x \in [0, 1], \quad t < 0,
\end{aligned} \tag{3.28}$$

where the residues are given by (3.27). Note that  $u_F$  is a locally integrable, real-valued function, which can shown similarly as in §3.1.1.

Therefore, in the case when boundary condition takes the form (3.3), the solution to system (2.3), (2.4), (2.5) reads

$$u(x, t) = u_H(x, t) + u_F(x, t), \quad x \in [0, 1], \quad t > 0,$$

where  $u_H$  and  $u_F$  are given by (3.18) and (3.28), respectively. Note that  $u_H$  and  $u_F$  are equal to zero for  $t < 0$ . Again, we have that  $u$  is a smooth function for  $x \in [0, 1], t > 0$ .

### 3.2 Determination of stress $\sigma$ in case of stress relaxation

We see that  $\tilde{T}$ , given by (2.13), has the branch point at  $s = 0$  and poles at the same points as  $\tilde{P}$ . Therefore, the poles of  $\tilde{T}$  are given as solutions to (3.4). Using the Cauchy residues theorem

$$\oint_\Gamma \tilde{T}(x, s) e^{st} ds = 2\pi i \sum_{n=1}^\infty \left[ \text{Res} \left( \tilde{T}(x, s) e^{st}, {}_P s_n^{(+)} \right) + \text{Res} \left( \tilde{T}(x, s) e^{st}, {}_P s_n^{(-)} \right) \right], \tag{3.29}$$

where contour  $\Gamma$  is given in figure 3.1, we obtain  $T$  in the following way. Residues in (3.29) are given by

$$\text{Res} \left( \tilde{T}(x, s) e^{st}, {}_P s_n^{(\pm)} \right) = \left[ \frac{\cosh(xsM(s))}{M(s) \frac{d}{ds} [\sinh(sM(s))]} e^{st} \right]_{s={}_P s_n^{(\pm)}}, \tag{3.30}$$

where  ${}_P s_n^{(\pm)}$ ,  $n \in \mathbb{N}$ , are solutions of (3.4).

Evaluating the integral at the left hand side of (3.29) in the same way as in §3.1.1, we obtain

$$\begin{aligned}
T(x, t) &= 1 + \frac{1}{2\pi i} \int_0^\infty \left( \frac{\cosh(xqM(qe^{i\pi}))}{M(qe^{i\pi}) \sinh(qM(qe^{i\pi}))} \right. \\
&\quad \left. - \frac{\cosh(xqM(qe^{-i\pi}))}{M(qe^{-i\pi}) \sinh(qM(qe^{-i\pi}))} \right) e^{-qt} dq \\
&\quad + \sum_{n=1}^\infty \left[ \text{Res} \left( \tilde{T}(x, s) e^{st}, p s_n^{(+)} \right) \right. \\
&\quad \left. + \text{Res} \left( \tilde{T}(x, s) e^{st}, p s_n^{(-)} \right) \right], \quad x \in [0, 1], \quad t > 0, \\
T(x, t) &= 0, \quad x \in [0, 1], \quad t < 0,
\end{aligned}$$

where the residues are given by (3.30). The proof is analogue to the one presented in §3.1.1.

Thus, by (2.14), we have

$$\sigma_H(x, t) = \Upsilon_0 T(x, t), \quad x \in [0, 1], \quad t > 0, \quad (3.31)$$

if boundary conditions (2.5) are given by (3.2). Note that  $\sigma_H$  is a locally integrable function with the jump at  $t = 0$  and smooth for  $t > 0$ . Also in case when boundary conditions (2.5) are given by (3.3), we have ( $x \in [0, 1]$ )

$$\begin{aligned}
\sigma_F(x, t) &= \sigma_H(x, t) + \frac{d}{dt} (F(t) * T(x, t)), \quad t > 0 \\
\sigma_F(x, t) &= 0, \quad t < 0.
\end{aligned}$$

This is a smooth function for  $t > 0$ . Note that  $\sigma_H$  and  $\sigma_F$  are real-valued functions, which can be shown similarly as in §3.1.1.

## 4 Numerical examples for displacement and stress in case of stress relaxation

In this section, we give several numerical examples of displacement  $u_H$  and stress  $\sigma_H$ , given by (3.18) and (3.31), respectively. In figure 4.1, we show displacements, determined according to (3.18), for two different positions. Parameters in (3.18) are chosen as follows:  $\Upsilon_0 = 1$ ,  $a = 0.045$ ,  $b = 0.5$ . The integration goes to 1000, while the number of residues in the sum is 400.

In figure 4.2, we show the stresses determined according to (3.31) for the same values of parameters used for figure 4.1. In order to emphasize stress relaxation process, we show stresses only for  $t \geq 1$ . From figure 4.2, it is evident that the stress tends to a constant value  $\Upsilon_0$ , at each point  $x \in [0, 1]$ .

## 5 Conclusion

In this work we analyze displacements  $u$  and stresses  $\sigma$  for a viscoelastic rod of finite length, which satisfy a constitutive equation of distributed fractional order (1.4), (1.5). Displacement of a free

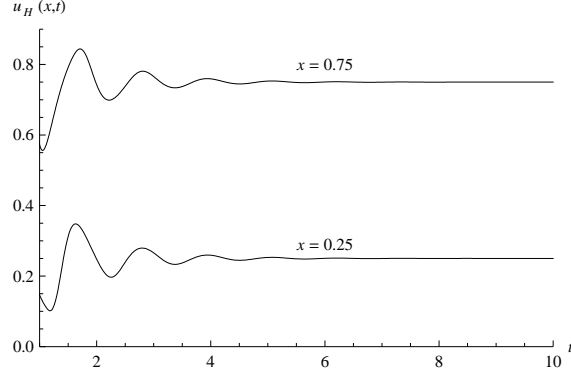


Figure 4.1: Displacements  $u_H(x, t)$  in stress relaxation experiment, as a function of time at  $x = 0.25$ ,  $x = 0.75$  for  $t \in (1, 10)$ .

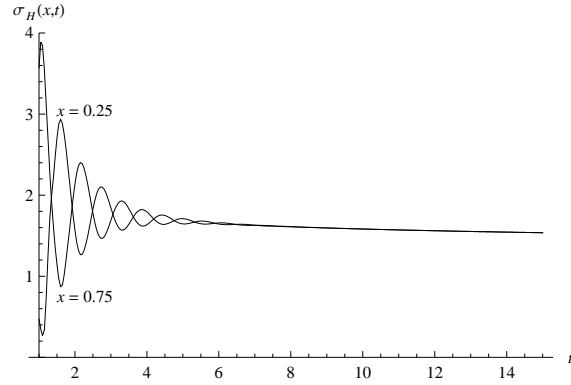


Figure 4.2: Stresses  $\sigma_H(x, t)$  in stress relaxation experiment, as a function of time at  $x = 0.25$ ,  $x = 0.75$  for  $t \in (1, 15)$ .

end of a rod is assumed to be  $\Upsilon = \Upsilon_0 H$  ( $\Upsilon_0 > 0$ ) and the displacement  $u_H$  is obtained in the form (3.18). Results for displacements and stresses are shown in figures 4.1 and 4.2. Figures show oscillatory character of both stresses and displacements. Oscillations are damped and for large time displacements show linear dependence on the distance of a particle from a fixed end, while stresses are approaching to the limiting value independently of the position of the particle. For large time stress relaxation curves tend to curves corresponding to quasistatic analysis (see work by Atanackovic 2002a and Drozdov 1998). In figures 4.1 and 4.2, we show displacements and stresses for  $t \geq 1$ .

Finally, let us comment the choice of parameters in constitutive equation (1.4), (1.5). Our choice  $a < b$  in (1.5) is a result of the requirement that entropy inequality is satisfied for each  $\alpha \in (0, 1)$  (see a paper by Bagley & Torvik 1986). The constitutive equation (1.4), (1.5) is of a viscoelastic type. A generalization of the problem treated here would include effects of viscoinertial type. In

that case (1.4) would be replaced by

$$\int_0^2 \phi_1(\alpha) {}_0D_t^\alpha \sigma(x, t) d\alpha = E \int_0^2 \phi_2(\alpha) {}_0D_t^\alpha \mathcal{E}(x, t) d\alpha, \quad x \in [0, L], \quad t > 0. \quad (5.1)$$

For the analysis of a system (1.1), (1.3), (5.1), one could use the type of analysis presented in work by Atanackovic *et al.* (2009b) and Atanackovic *et al.* (2009c).

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